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An analytical VLBI delay formula for Earth satellites

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Geodetic VLBI observations of Earth satellites

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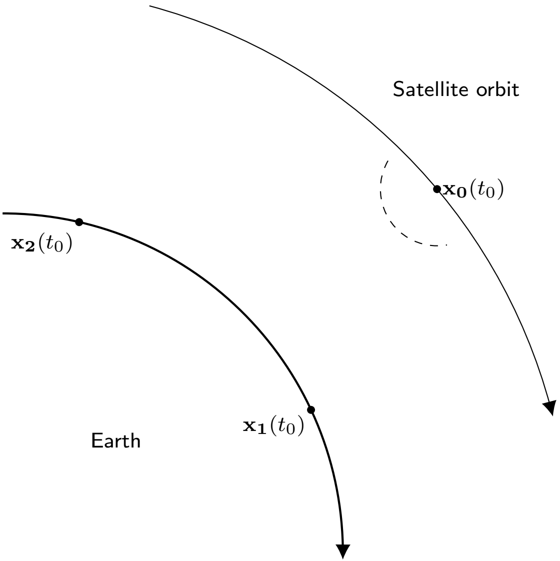
- Suitable modelling of the VLBI delay → This talk. . .

Signal propagation



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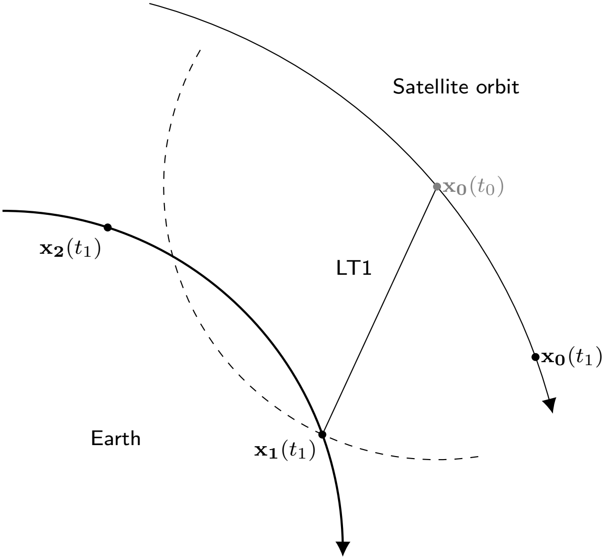
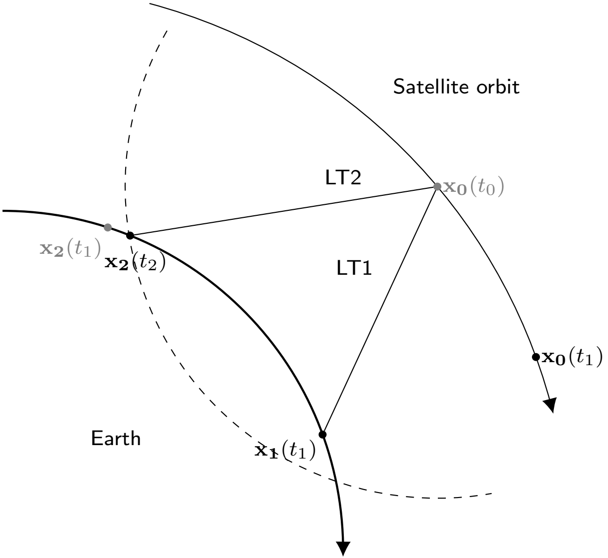
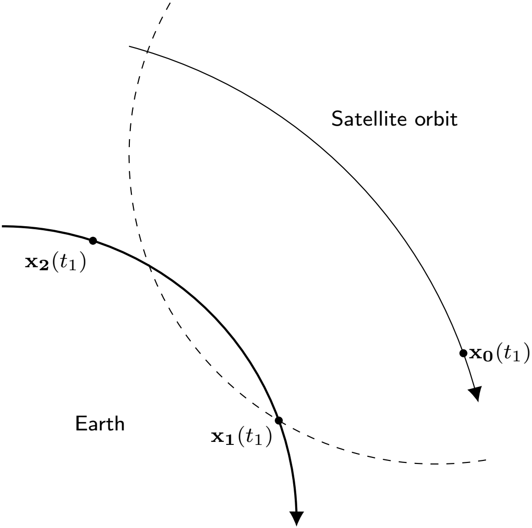



Satellite orbit
LT2
$\mathbf{x_0}(t_0)$
$\mathbf{x_2}(t_1)$
$\mathbf{x_2}(t_2)$
LT1
$\mathbf{x_0}(t_1)$
Earth
$\mathbf{x_1}(t_1)$

Geometry at reception time t_1

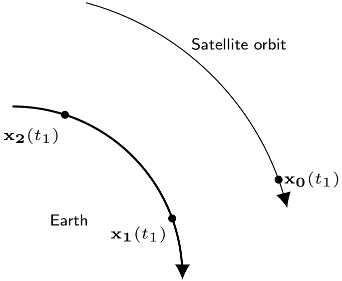


Solving the light-time-equations

1. To find t_0 and $\mathbf{x}_0(t_0)$, vary t_0 until

$$t_0 = t_1 - \frac{|\mathbf{x}_1(t_1) - \mathbf{x}_0(t_0)|}{c} - t_{g01}$$

is fulfilled to desired precision.



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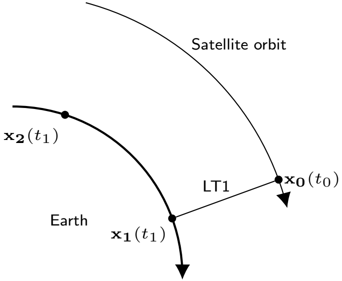

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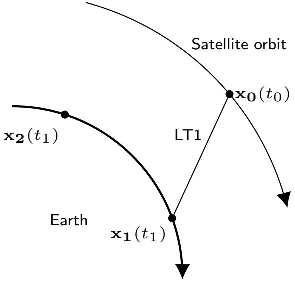

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2. To find t_2 and $\mathbf{x}_2(t_2)$, vary t_2 until

$$t_2 = t_0 + \frac{|\mathbf{x}_2(t_2) - \mathbf{x}_0(t_0)|}{c} + t_{\mathrm{g02}}$$

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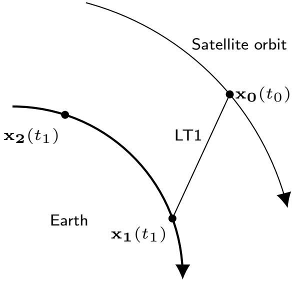

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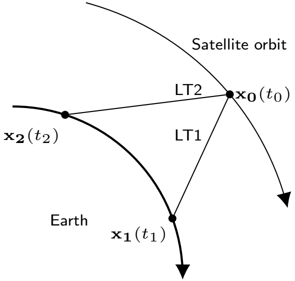

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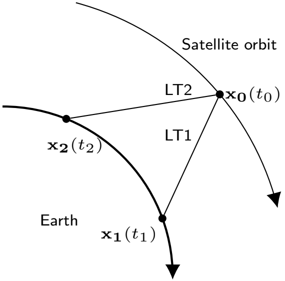

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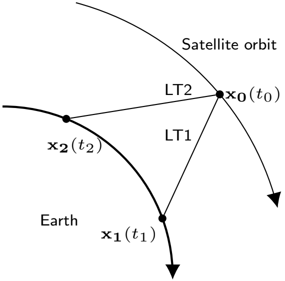
is fulfilled to desired precision.

3. $\tau = (t_2 - t_1)(1 - L_G)$ (in TT)
with $L_G = 6.969290134 \cdot 10^{-10}$
Kaplan 2005, U.S. Naval Observatory Circular, No. 179



Solving the light-time-equation

Usual approach: Numerical method (e.g., Newton-Raphson)

Sekido & Fukushima, 2006, JGeod, 80, 3

Duev *et al.*, 2012, A&A, 541, A43

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Can the light-time-equation be solved

- analytically
- linearizing the problem?

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Can the light-time-equation be solved

- analytically
- linearizing the problem?

→ **Yes!**

The difference in signal arrival times, in GCRS, is

$$\tau_{\mathrm{TGC}} = t_2 - t_1 = t_2 - t_0 + t_0 - t_1 = \Delta t_2 + \Delta t_0,$$

Transforming to the terrestrial time,

$$\tau_{\mathrm{TT}} = (\Delta t_2 + \Delta t_0)\,(1 - L_G)\,,$$

with

$$\begin{aligned}\Delta t_0 \quad &= \quad \gamma_0^2 \left[\frac{\vec{x}_{01}\cdot\vec{v}_0}{c^2} - t_{\mathrm{g}\,01}\right] \\ &\quad - \sqrt{\gamma_0^4 \left[\frac{\vec{x}_{01}\cdot\vec{v}_0}{c^2} - t_{\mathrm{g}\,01}\right]^2 + \gamma_0^2 \left[\frac{x_{01}^2}{c^2} - t_{\mathrm{g}\,01}^2\right]},\end{aligned}$$

and

$$\begin{aligned}\Delta t_2 \quad &= \quad \gamma_2^2 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02}\cdot\vec{v}_2}{c^2}\right] \\ &\quad + \sqrt{\gamma_2^4 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02}\cdot\vec{v}_2}{c^2}\right]^2 + \gamma_2^2 \left[\frac{x_{02}^2}{c^2} - t_{\mathrm{g}\,02}^2\right]}.\end{aligned}$$

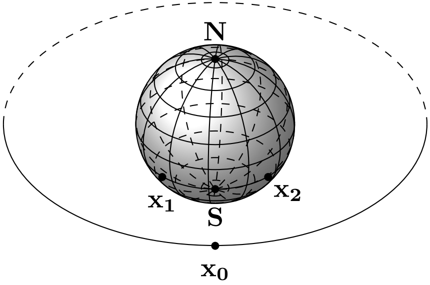
For details see Jaron & Nothnagel, 2018, JGeod .

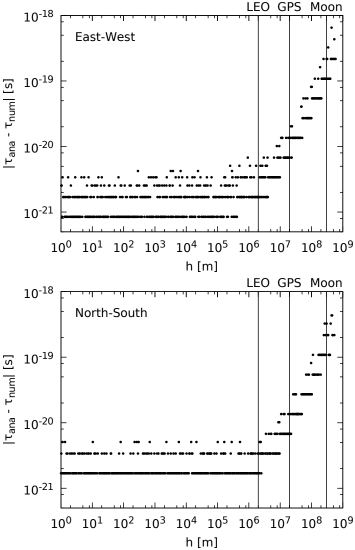
Testing the model

Differences between analytical and numerical solution?

- How large?
- Any systematics?

Testing the model





- Satellite $\mathbf{x}_0$ at a configurable altitude h, circular orbit, $\omega_{\text{sat}} = \sqrt{\frac{GM}{(h+R)^3}}$.
- Two stations at $\mathbf{x}_1$ and $\mathbf{x}_2$ on surface of rotating Earth.

Applicability to GRASP and E-GRASP

Proposed co-location of SLR, GNSS, DORIS, and VLBI, in space:

- GRASP $e = 0.03$ Bar-Sever *et al.* 2009
- E-GRASP $e = 0.3$ Biancale *et al.* 2017

Kepler's equation: $E = M + e \sin E$

Mean anomaly: $M = M_0 + n(t - \tau)$

Angular velocity: $n = \sqrt{\frac{GM_{\text{Earth}}}{a^3}}$

Observing perigee passage 10^5 times with random baselines results in:

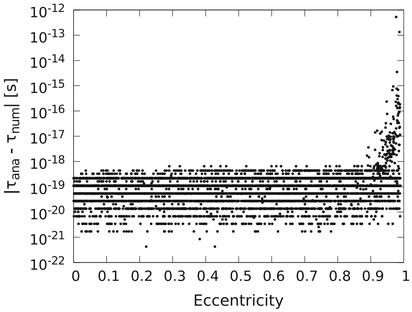
$< |\text{analytical - numerical}| >$

GRASP: $(1.6 \pm 1.5)\, 10^{-21}$ s

E-GRASP: $(2.3 \pm 2.0)\, 10^{-21}$ s

Simulating 10^6 random observations

Dependency on eccentricity



$$d = \frac{1}{N}\sum_{i=1}^{N} \left|\tau_{\text{ana},i} - \tau_{\text{num},i}\right|$$

d [s]	e-range
$(7.3 \pm 14.1)\,10^{-20}$	$0.0 - 0.9$
$(1.6 \pm 358)\,10^{-15}$	$0.9 - 1.0$
$(1.5 \pm 108)\,10^{-16}$	$0.0 - 1.0$

$$|\text{analytical} - \text{numerical}| < 1\,\text{ps}$$

Conclusions

1. Light-time equation has an analytical solution when linearized.
2. For Earth satellites: differences between numerical and analytical solution is way below the detection limit of VLBI.
3. Analytical delay formula implies analytical partial derivatives.

Jaron & Nothnagel, 2018, JGeod

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Thank you!

Appendix

Linearize satellite orbit around t_1,

$$\vec{x}_{0,\,\mathrm{lin}}(t) = \vec{x}_0(t_1) + \vec{v}_0(t_1) \cdot [t - t_1]. \tag{1}$$

Analytical solution for t_0

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Shift time-axis such that $t_1 = 0$,

$$\vec{x}_{0,\,\mathrm{lin}}(t) = \vec{x}_0 + \vec{v}_0 \cdot t. \tag{2}$$

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Light-time-equation for t_0,

$$t_0 = -\frac{|\vec{x}_0(t_0) - \vec{x}_1|}{c} - t_{\mathrm{g}\,01}. \tag{3}$$

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Inserting (2) into (3) yields

$$t_0 = -\frac{|\vec{x}_0 + \vec{v}_0 \cdot t_0 - \vec{x}_1|}{c} - t_{\mathrm{g}\,01}. \tag{4}$$

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$$\Rightarrow \left[t_0 + t_{\mathrm{g}\,01}\right]^2 = \frac{|\vec{x}_0 + \vec{v}_0 \cdot t_0 - \vec{x}_1|^2}{c^2} \tag{5}$$

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Quadratic equation in t_0 with the two formal solutions

$$\begin{aligned} t_0 \quad &= \quad \gamma_0^2 \left[\frac{\vec{x}_{01} \cdot \vec{v}_0}{c^2} - t_{\mathrm{g}\,01} \right] \\ &\pm \sqrt{\gamma_0^4 \left[\frac{\vec{x}_{01} \cdot \vec{v}_0}{c^2} - t_{\mathrm{g}\,01} \right]^2 + \gamma_0^2 \left[\frac{x_{01}^2}{c^2} - t_{\mathrm{g}\,01}^2 \right]}, \end{aligned} \qquad (6)$$

with $\vec{x}_{01} = \vec{x}_0 - \vec{x}_1$ and $\gamma_0^2 = \left(1 - v_0^2/c^2\right)^{-1}$.

Analytical solution for t_0

$$t_0 = -\frac{|\vec{x}_0 + \vec{v}_0 \cdot t_0 - \vec{x}_1|}{c} - t_{\mathrm{g}\,01}. \tag{4}$$

$$\Rightarrow [t_0 + t_{\mathrm{g}\,01}]^2 = \frac{|\vec{x}_0 + \vec{v}_0 \cdot t_0 - \vec{x}_1|^2}{c^2} \tag{5}$$

Since t_0 has to be negative, the correct solution is

$$\begin{aligned} t_0 \quad = \quad & \gamma_0^2 \left[\frac{\vec{x}_{01} \cdot \vec{v}_0}{c^2} - t_{\mathrm{g}\,01} \right] \\ & - \sqrt{\gamma_0^4 \left[\frac{\vec{x}_{01} \cdot \vec{v}_0}{c^2} - t_{\mathrm{g}\,01} \right]^2 + \gamma_0^2 \left[\frac{x_{01}^2}{c^2} - t_{\mathrm{g}\,01}^2 \right]}, \end{aligned} \tag{6}$$

with $\vec{x}_{01} = \vec{x}_0 - \vec{x}_1$ and $\gamma_0^2 = \left(1 - v_0^2/c^2\right)^{-1}$.

Analytical solution for t_2

Linearize position of station 2 around t_1,

$$\vec{x}_{2,\,\mathrm{lin}}(t) = \vec{x}_2(t_1) + \vec{v}_2(t_1) \cdot [t - t_1] \overset{t_1=0}{=} \vec{x}_2 + \vec{v}_2 t. \tag{7}$$

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Light-time-equation for t_2,

$$t_2 = t_0 + \frac{|\vec{x}_0(t_0) - \vec{x}_2(t_2)|}{c} + t_{\mathrm{g}\,02}. \qquad (8)$$

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Introducing the light-travel time $\Delta t_2 = t_2 - t_0$,

$$\Delta t_2 = \frac{|\vec{x}_0(t_0) - \vec{x}_2(\Delta t_2 + t_0)|}{c} + t_{\mathrm{g}\,02}. \tag{9}$$

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Inserting (2) and (7) into (9) yields

$$\Delta t_2 = \frac{|\vec{x}_0 + \vec{v}_0 t_0 - [\vec{x}_2 + \vec{v}_2[\Delta t_2 + t_0]]|}{c} + t_{\mathrm{g}\,02} \tag{10}$$

$$\Delta t_2 = \frac{|\vec{x}_0 + \vec{v}_0 t_0 - [\vec{x}_2 + \vec{v}_2[\Delta t_2 + t_0]]|}{c} + t_{\mathrm{g}\,02} = \frac{|\vec{x}_{02}(t_0) - \vec{v}_2 \Delta t_2|}{c} + t_{\mathrm{g}\,02}, \tag{11}$$

with $\vec{x}_{02} = \vec{x}_0 - \vec{x}_2 + [\vec{v}_0 - \vec{v}_2]t_0$.

Analytical solution for t_2

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with $\vec{x}_{02} = \vec{x}_0 - \vec{x}_2 + [\vec{v}_0 - \vec{v}_2]t_0$.

$$(11) \Rightarrow [\Delta t_2 - t_{\mathrm{g}\,02}]^2 = \frac{[\vec{x}_{02} - \vec{v}_2 \Delta t_2]^2}{c^2} \tag{12}$$

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with the two formal solutions

$$\begin{aligned}\Delta t_2 &= \gamma_2^2 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02} \cdot \vec{v}_2}{c^2} \right] \\ &\quad \pm \sqrt{\gamma_2^4 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02} \cdot \vec{v}_2}{c^2} \right]^2 + \gamma_2^2 \left[\frac{x_{02}^2}{c^2} - t_{\mathrm{g}\,02}^2 \right]}, \qquad (13)\end{aligned}$$

where $\gamma_2^2 = \left(1 - v_2^2/c^2\right)^{-1}$.

Analytical solution for t_2

$$\Delta t_2 = \frac{|\vec{x}_0 + \vec{v}_0 t_0 - [\vec{x}_2 + \vec{v}_2[\Delta t_2 + t_0]]|}{c} + t_{\mathrm{g}\,02}$$
$$= \frac{|\vec{x}_{02}(t_0) - \vec{v}_2 \Delta t_2|}{c} + t_{\mathrm{g}\,02}, \quad (11)$$

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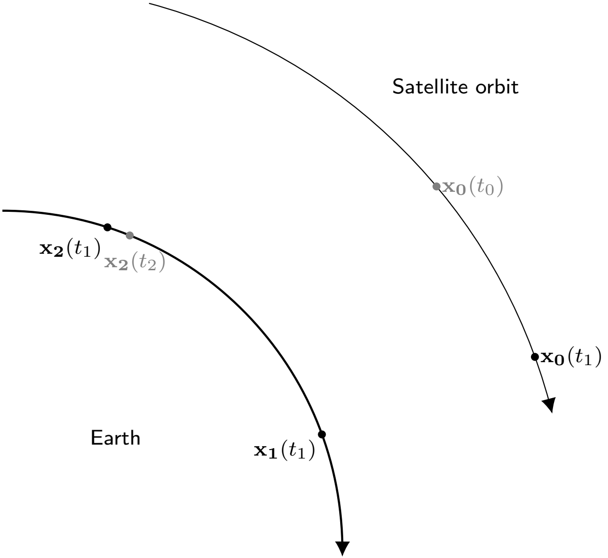
$$(11) \Rightarrow [\Delta t_2 - t_{\mathrm{g}\,02}]^2 = \frac{[\vec{x}_{02} - \vec{v}_2 \Delta t_2]^2}{c^2} \quad (12)$$

Since the light travel time Δt_2 is positive,

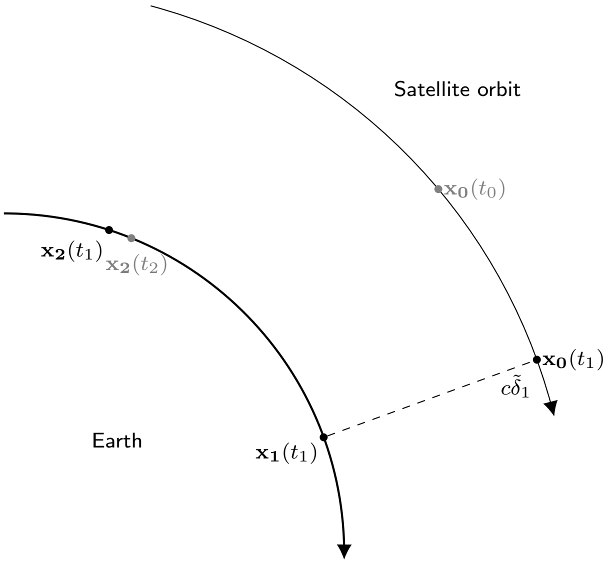
$$\Delta t_2 = \gamma_2^2 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02} \cdot \vec{v}_2}{c^2} \right] + \sqrt{\gamma_2^4 \left[t_{\mathrm{g}\,02} - \frac{\vec{x}_{02} \cdot \vec{v}_2}{c^2} \right]^2 + \gamma_2^2 \left[\frac{x_{02}^2}{c^2} - t_{\mathrm{g}\,02}^2 \right]}, \quad (13)$$

where $\gamma_2^2 = (1 - v_2^2/c^2)^{-1}$.

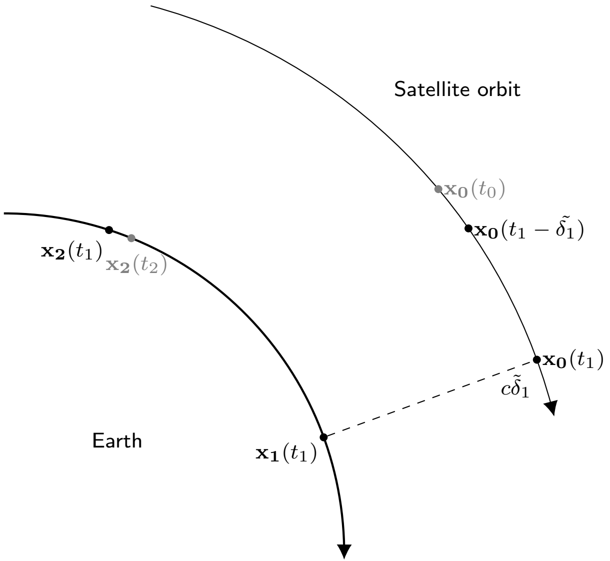
Choosing the tangent points



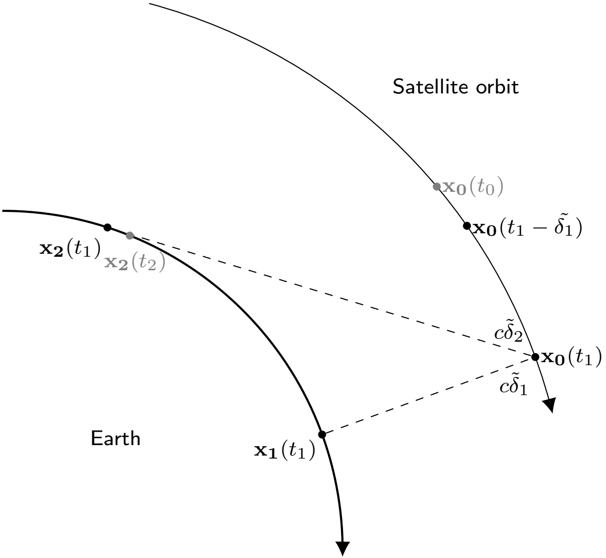
Choosing the tangent points



Choosing the tangent points



Choosing the tangent points



Choosing the tangent points