22. EVGA Meeting 2015

Numerical Issues of VLBI Data Analysis

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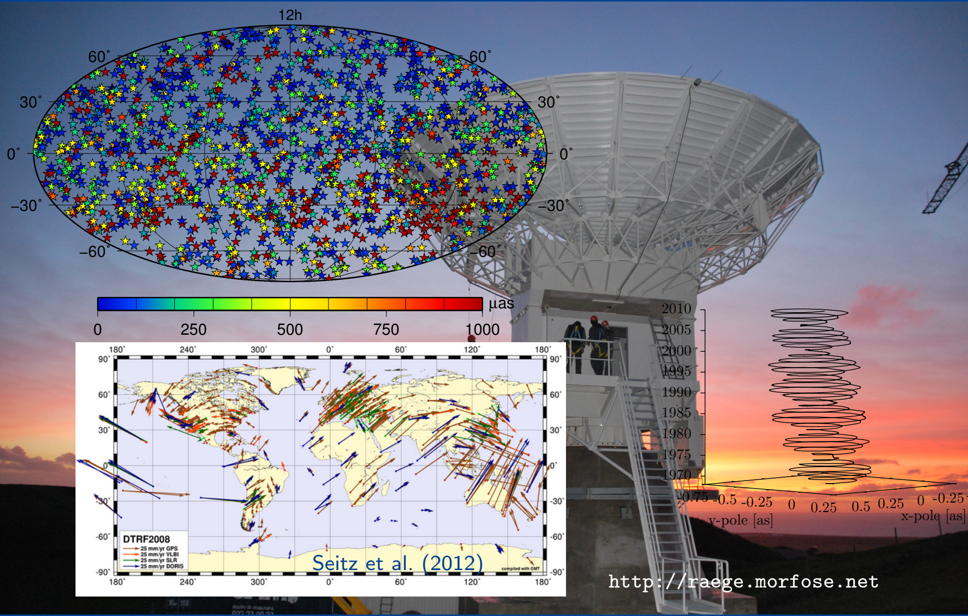
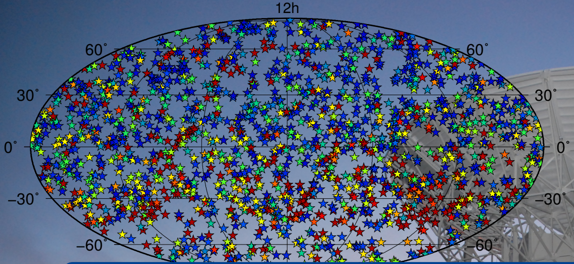
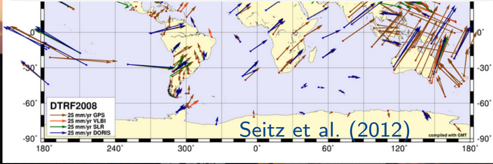
VLBI

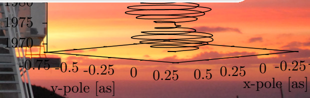

12h
60°
30°
0°
−30°
−60°
µas
0
250
500
750
1000
DTRF2008
25 mm/yr GPS
25 mm/yr VLBI
25 mm/yr SLR
25 mm/yr DORIS
Seitz et al. (2012)
2010
2005
2000
1995
1990
1985
1980
1975
1970
y-pole [as]
x-pole [as]
http://raege.morfose.net



Parameter Estimation

Least squares adjustment:
gradient of $\|\mathbf{Ax}-\mathbf{b}\|_2^2 = \langle \mathbf{Ax}-\mathbf{b}\rangle$ vanishes
$\Rightarrow 0 = \mathbf{A}^T\mathbf{r} = \mathbf{A}^T\mathbf{Ax} - \mathbf{A}^T\mathbf{b}$





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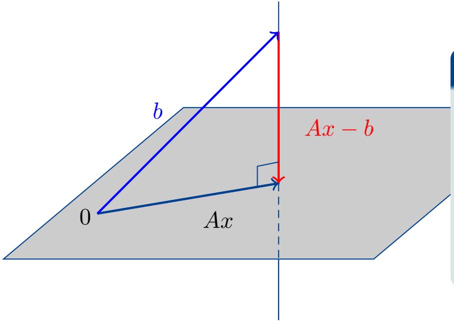
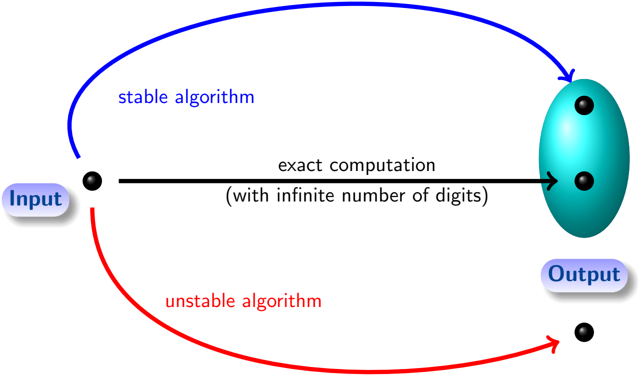


Numerical Issues

- Condition of the Jacobian matrix
  $\Rightarrow$ orientation of the plane
- stability of the algorithm
  $\Rightarrow$ precision of mapping



Stability of the Algorithm

If an algorithm is numerically unstable, at a given point, the errors do not remain bounded and tend to grow up in an uncontrolled way corrupting completely the final result.

LS Solution Algorithms

Normal Equations

$$\mathbf{N} = \mathbf{A}^T\mathbf{A}, \;\; \mathbf{n} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{n}$$

$\mathbf{N}$: symmetric & positive definite

$\Rightarrow$ Cholesky decomposition

$$\mathbf{A} = \mathbf{L}^T\mathbf{L}$$

and back solution to determine $\mathbf{x}$.

QR Decomposition

Decompose $\mathbf{A}$ in orthogonal matrix $\mathbf{Q}$ and upper tri-angular matrix $\mathbf{R}$

$$\mathbf{A} = \mathbf{QR} \;\; \Rightarrow \mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}$$

SVD

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T.$$

$\Rightarrow$ solution of $min_{\mathbf{x}}\|\mathbf{A}\cdot\mathbf{x} - \mathbf{b}\|_2$:

$$\mathbf{x} = \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^T\mathbf{b}.$$

LS Solution Algorithms

Normal Equations

$$\mathbf{N} = \mathbf{A}^T\mathbf{A}, \quad \mathbf{n} = \mathbf{A}^T\mathbf{b}$$

fast, but least accurate

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{n}$$

$\mathbf{N}$: symmetric & positive definite
$\Rightarrow$ Cholesky decomposition

$$\mathbf{A} = \mathbf{L}^T\mathbf{L}$$

and back solution to determine $\mathbf{x}$.

QR Decomposition

Decompose $\mathbf{A}$ in orthogonal matrix $\mathbf{Q}$ and upper tri-angular matrix $\mathbf{R}$

costs up to twice as NEQ

$$\mathbf{A} = \mathbf{QR} \quad \Rightarrow \mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}$$

SVD

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$$

for ill-conditioned systems;
$\mathbf{A}$ is singular

$\Rightarrow$ solution of $min_{\mathbf{x}} \|\mathbf{A}\cdot\mathbf{x} - \mathbf{b}\|_2$:

$$\mathbf{x} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T\mathbf{b}.$$

Weighted Solution

$$\mathbf{N} = \mathbf{A}^T\mathbf{W}\mathbf{A} = \mathbf{A}\Sigma_{bb}^{-1}\mathbf{A}^T$$

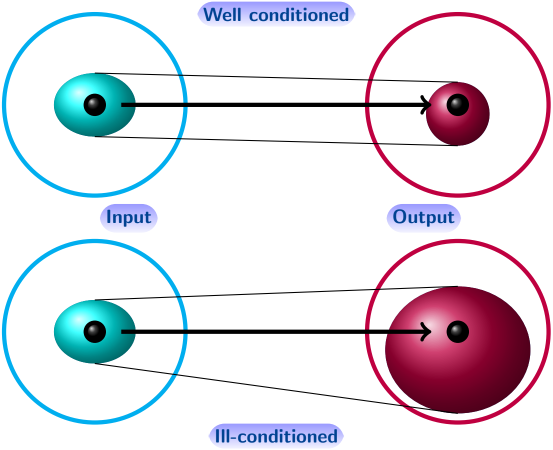
$$\mathbf{n} = \mathbf{A}^T\mathbf{W}\mathbf{b}$$

De-correlation

$$\mathbf{W} = \mathbf{L}^T\mathbf{L}$$

$$\widetilde{\mathbf{A}} = \mathbf{L}\mathbf{A}, \widetilde{\mathbf{b}} = \mathbf{L}\mathbf{b}$$

$$\begin{aligned}\mathbf{x} &= \left(\widetilde{\mathbf{A}}^T\mathbf{E}\widetilde{\mathbf{A}}\right)^{-1}\widetilde{\mathbf{A}}^T\mathbf{E}\widetilde{\mathbf{b}} \\ &= \left(\mathbf{A}^T\mathbf{L}^T\mathbf{L}\mathbf{A}^T\right)^{-1}\mathbf{A}^T\mathbf{L}^T\mathbf{L}\mathbf{b} \\ &= \left(\mathbf{A}^T\mathbf{W}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b}\end{aligned}$$

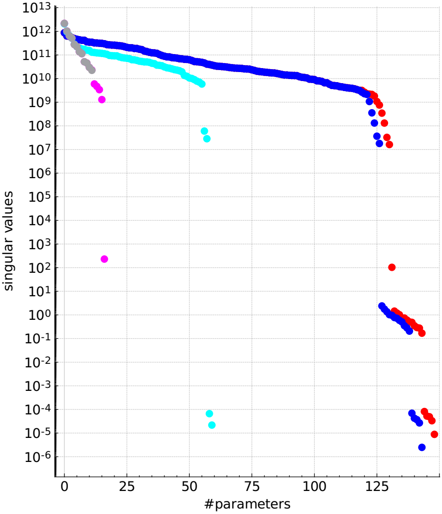


Well Conditioned

a small perturbation of the input data leads to small variations of the results

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

$$\kappa(\mathbf{A}) = \frac{\sigma_{max}}{\sigma_{min}}$$



CL	ZWD	GR	ERP	NUT
d2	d1	-	-	-
d2	d1	-	d0	d0
d2 c1	d1	-	-	-
d2 c1	c1	c24	-	-
d2 c1	c1	c24	d0	d0

d: polynomial of degree;

c: CPWLF with interval length [h]

$\kappa(\mathbf{A})$	rank
$9.7 \cdot 10^{1}$	
$9.4 \cdot 10^{9}$	
$9.6 \cdot 10^{16}$	
$3.5 \cdot 10^{17}$	
$9.8 \cdot 10^{16}$	

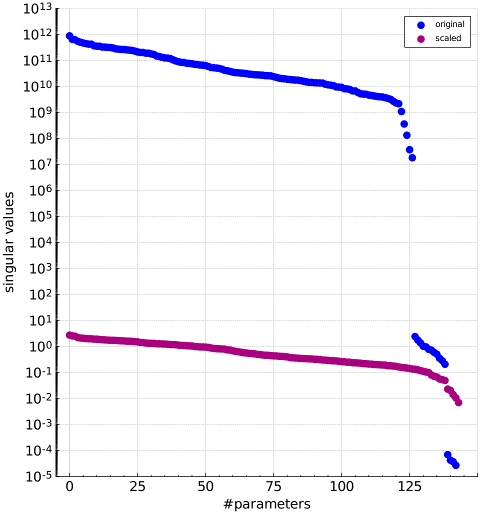
Scaling

$$\hat{\mathbf{A}} = \widetilde{\mathbf{A}}\mathbf{F}$$

$$\mathbf{F} = diag\left(\frac{1}{\sqrt{diag\left(\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}}\right)}}\right)$$

$\Rightarrow$ only 1 on main diagonal of $\hat{\mathbf{N}}$

$\kappa(\mathbf{A})$	rank
$3.5 \cdot 10^{17}$	
$3.9 \cdot 10^{2}$	



Change $\kappa(\mathbf{A})$ – Scaling

Scaling

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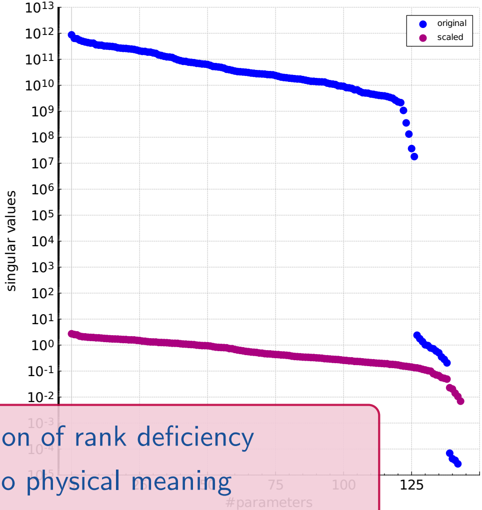
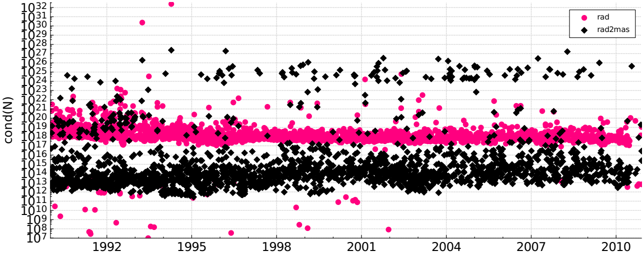


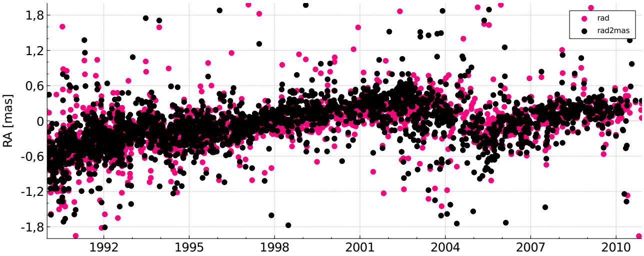
- elimination of rank deficiency
- units w/o physical meaning
- estimates have to be re-scaled: $\mathbf{x} = \hat{\mathbf{x}}\mathbf{F}$

Change $\kappa(\mathbf{A})$ – Units

Scaling

- gsf2014a
- SINEX: rad
- changed to mas





- $\kappa(\mathbf{N})$ typically improves
- different results
- same noise level

Constraining

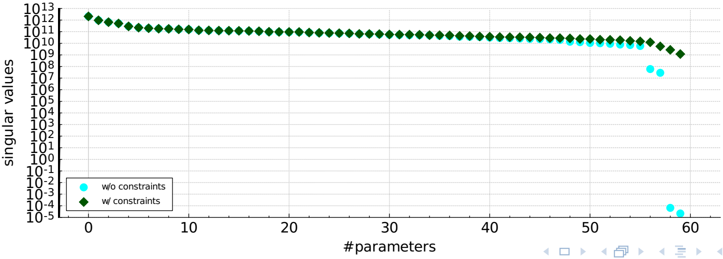
Add information, e.g., rate between two subsequent ZWD is zero with a given σ_{Bi}

$$\mathbf{B} = \begin{pmatrix} 0 & \dots & 1 & -1 & 0 & \dots \end{pmatrix},$$

$$\mathbf{W}_B = diag(1/{\sigma_{Bi}}^2)$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_B \end{pmatrix}$$

$\kappa(\mathbf{A})$	rank
$9.6 \cdot 10^{16}$	
$1.7 \cdot 10^{3}$	



Constraining

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

$$\mathbf{\Sigma} = diag\left(\sigma_1 \quad \sigma_2 \quad \dots \quad \sigma_r \quad 0 \quad \dots \quad 0\right)$$

$$\mathbf{U} = \Big(\underbrace{u_1 \quad u_2 \quad \dots \quad u_r}_{\mathbf{U}_r} \quad \underbrace{u_{r+1} \quad \dots \quad u_m}_{\mathbf{U}_0} \Big)$$

$$\mathbf{V} = \Big(\underbrace{v_1 \quad v_2 \quad \dots \quad v_r}_{\mathbf{V}_r} \quad \underbrace{v_{r+1} \quad \dots \quad v_n}_{\mathbf{V}_0} \Big)$$

Add information, e.g., rate between two subsequent ZWD is zero with a given σ_{Bi}

$$\mathbf{B} = \left(0 \ \dots \ 1 \ -1 \ 0 \ \dots\right);$$

$$\mathbf{W}_B = diag(1/\sigma_{Bi}{}^2)$$

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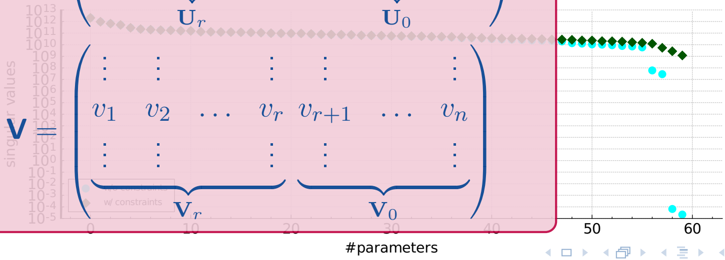
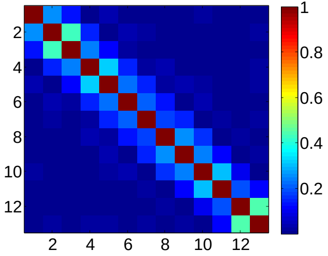
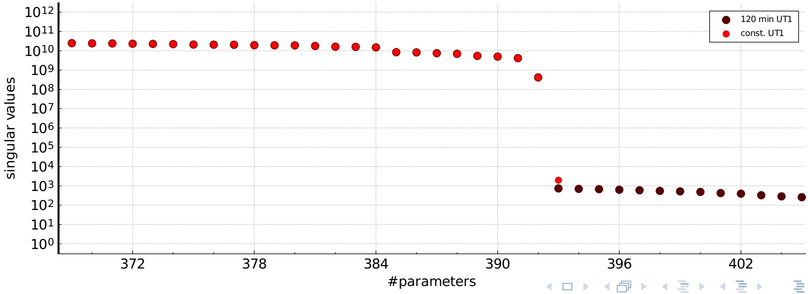


UT1 at Finer Intervals



Analysis Workshop

What happens if we solve for UT1 at finer intervals?



UT1 at Finer Intervals

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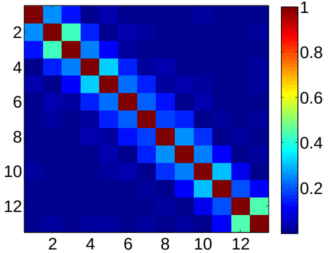

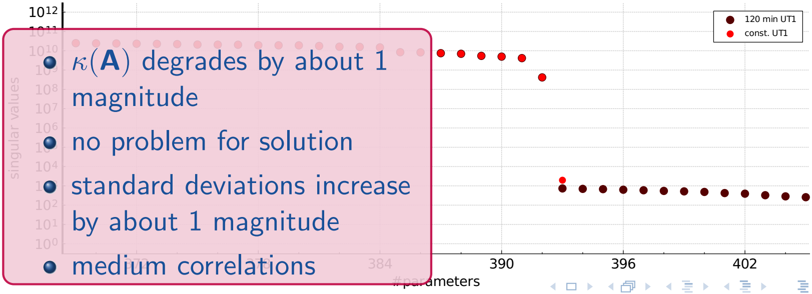

- $\kappa(\mathbf{A})$ degrades by about 1 magnitude
- no problem for solution
- standard deviations increase by about 1 magnitude
- medium correlations

Conclusions & Outlook

- equation system ill-conditioned
- simple options to improve condition
- different results with different modifications of $\kappa(\mathbf{A})$
- which is the correct approach?

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possible next steps

- optimal selection of constraints
- investigation of parameterization options
- validation of algorithms

Thank you!

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